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THE NON-CONSTANCY OF THE BLACK-SCHOLES VARIANCE PARAMETER: A GENERALISED METHOD OF MOMENTS INVESTIGATION

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ABSTRACT

This paper considers the non-constant nature of the volatility parameter in the Black and Scholes (1973) option pricing model. References in the literature to the many empirical observations of systematic pricing biases associated with the model, in particular the exercise price and the time-to-maturity biases, are widespread. London International Financial Futures and Options Exchange (LIFFE) close-of-day data on the FTSE 100 European-style exercise stock index-option (ESX) is employed to test the Black-Scholes model specification using Hansen's (1982) Generalised Method of Moments (GMM). The results of the empirical analysis confirm the existence of pricing biases and might be interpreted as indirect evidence in support of the hypothesis that the market is using an alternative model to price options.

INTRODUCTION

When one calculates option values using the Black-Scholes model and compares them with option prices, there is usually a discrepancy. It is rare that the theoretical value of an option is exactly equal to the price at which it is traded on the exchange. Since the derivation of the Black-Scholes option pricing formula, many empirical studies that focus on its ability to price options have appeared in the literature. It is well documented that the theoretical model does induce price biases. Hence, despite its undoubted contribution to the world of finance, the model does contain anomalies that limit its ability to identify under- or over-priced options.

The existence of such biases suggests that the Black-Scholes model may be incorrectly specified in relation to the true underlying data generation process. In particular, researchers have focused on the assumption that the volatility of stock price returns is a constant. Given the weight of evidence contradicting the constant volatility assumption, several models that allow volatility to vary either deterministically or according to some independent stochastic process have appeared in the literature. The purpose of this paper is to investigate, and to confirm independently if at all possible, the well-documented non-constancy feature of the Black-Scholes variance parameter.

The paper is organised as follows. The next section briefly examines the biases associated with the Black-Scholes model and puts forward some possible explanations for these biases. The third section looks at the methodology for the empirical investigation of these biases. Section four describes the data. The results of the empirical analysis are presented in section five and the paper is summarised in the final section.

THEORY

In particular, researchers have focused upon two observed biases associated with the Black-Scholes option pricing model, the time-to-maturity bias and the exercise price or moneyness bias. For example, according to Macbeth (1979) and Merville (1980), the model tends to overprice in-the-money call options while it underprices out-of-the-money call options. Furthermore, Rubinstein (1985) finds that short-maturity out-of-the-money calls are priced significantly higher relative to other calls than the Black-Scholes model would predict. Other observed biases include the interest rate bias (Rindell, 1994) and the variance bias (Black and Scholes, 1972).

A number of explanations seek to account for the systematic price biases in Black-Scholes model option prices. In particular, the assumption of a lognormally distributed security price, which fails to systematically capture important characteristics of the true security price process, has been questioned (Ball and Torous, 1985). It is generally accepted that security price returns are not normally distributed and that observed distributions for security price returns and other asset returns are found to have tails fatter than are consistent with a normal distribution (Mandlebrot, 1963; Fama, 1965; Blattberg and Gonedes, 1974; Kon, 1984). It follows that the Black-Scholes model will tend to undervalue both out-of-the-money puts and out-of-the-money calls, because it will understate the chances of large positive or negative returns. Correspondingly, because of put-call parity, in-the-money calls will also be undervalued, and if out-of-the-money calls are undervalued so also will in-the-money puts (Gemmill, 1993). Therefore, the existence of an exercise price bias is perhaps less surprising when the true distribution of stock price returns is inconsistent with that assumed by Black and Scholes (1973).

Another possible source of bias (and one that is closely related to the above) is the specification of the stochastic movement of the stock price through time. The stock is assumed to follow a stationary diffusion process, which restricts the stock to smooth continuous price changes. Hence, Merton (1976) notes that the validity of the Black-Scholes formula depends on whether or not stock price changes satisfy a kind of local Markov property. Empirical data shows that this assumption is often violated, as stocks frequently exhibit sudden non-marginal price movements known as jumps.

Geske and Roll (1984, p. 454) offer a possible explanation for the variance bias. They suggest that it 'could be attributed to non-stationary stock volatility which could be induced by dividend uncertainty.' Trautman (1983), however, demonstrates that the consideration of this aspect does not remove the variance bias. By examining a data set of options on non-dividend paying stocks, he finds an even more significant variance bias than in an examination of a data set of options on both dividend and non-dividend paying stocks.

Following Bossaerts and Hillion (1989), Chan, Karolyi, Longstaff and Sanders (1991), Rindell (1994) and Day and Lewis (1997), our econometric approach will be to test a set of overidentifying restrictions on a system of moment equations using Generalised Method of Moments (GMM). Using such an approach, the pricing performance of the Black and Scholes (1973) option pricing formula is examined. In particular, the issue of whether the pricing errors under the Black and Scholes (1973) model are a systematic function of time-to-maturity, moneyness and the risk-free interest rate is investigated. More specifically, our objective will be to search for empirical evidence that the volatility function, or process, embedded in LIFFE FTSE 100 stock index European-style option (ESX) prices is not consistent with the constant-volatility parameterisation implied by the Black-Scholes model. If such a conclusion is in fact supported by the data, then assuming away market inefficiencies and non-synchronous measurement errors, one might reasonably conclude that either a more elaborate functional form, or perhaps an autonomous stochastic specification, would be required to characterise the volatility parameter and thus correctly account for the cross-sectional complex of ESX option prices that trade on LIFFE.

METHODOLOGY

Introduction

Consider that one might expect the differences between Black-Scholes model values and corresponding market prices to be zero on average, or perhaps to display no systematic behaviour in relation to any variable comprising the investor's information set, if the Black-Scholes model is indeed used by market participants in determining option prices. A GMM estimation procedure can be applied germanely to test the model's goodness-of-fit with market data by specifying the pricing errors, or their relationship with (instrumental) variables in the information set, as a family of moment conditions. More specifically, the GMM estimator is based on the minimisation of the distance of a vector of such moment conditions, characterised germanely as orthogonality conditions, from zero (Bossaerts and Hillion, 1989). This distance is minimised with respect to the parameters of the model in question, which are implied from the data rather than estimated from historical data.

When the number of distinct moment equations exceeds the number of parameters to be estimated (for example, if there are four moment equations and two unknown parameters to be estimated), any two of the four moment equations could be chosen to find a unique solution. However, all of the information in the sample is not being efficiently used by choosing only two of the four moment equations. Another issue is, of course, which two moment conditions should be chosen given that any pair out of the six possible combinations may produce different estimates of the parameter(s) being estimated. Therefore, it is necessary to devise a method to reconcile the conflicting estimates that will emerge from what is termed an overidentified system. The method used for this reconciliation is developed in Hansen's (1982) celebrated paper. Bossaerts and Hillion (1989) explain the reconciliation procedure as follows: they note that, when a system is overidentified and no exact solution to the system obtains, the problem can be solved by forming a vector composed of the sample moment conditions and minimising it with respect to the parameters being estimated, i.e., making each element in the vector as close as possible to zero.

GMM applications in the option pricing literature are not particularly widespread. However, Bossaerts and Hillion (1989) and Rindell (1994) perform tests on option pricing models using Hansen's (1982) GMM procedure. Bossaerts and Hillion (1989), in an examination of different versions of the Black (1976) commodity futures-option pricing model, test the joint hypothesis that: i) the contingent claims asset pricing model is correct, i.e., has the same functional form as that used by the market, ii) markets are non-synchronous, and iii) the data employed in the study is accurate. In a similar study, Rindell (1994) tests the hypothesis that the difference between market prices and theoretical prices given by the Black and Scholes (1973) stock option pricing model is a function of the time-to-maturity of the option, the extent to which the option is in the money and the current interest rate.

Elsewhere in the contingent claims literature, Chan et al. (1991) use a GMM procedure to estimate and compare a variety of single-factor term structure models. Other empirical tests of the term structure have also used GMM (Gibbons and Ramaswamy, 1986; Harvey, 1988; Longstaff, 1989; Pearson and Sun, 1989). Day and Lewis (1997) use GMM in their examination of the relationship between the volatility of the crude oil futures market and changes in initial margin requirements. The analysis and empirical framework that follows is most typical of that found in the papers of Bossaerts and Hillion (1989) and Rindell (1994), but is qualitatively also quite similar to that in the other papers cited.

Hypothesis Formulation

In order to test the dividend-adjusted Black-Scholes model using LIFFE data on the European-style FTSE 100 stock index-option contract, it is necessary first to formulate an appropriate null hypothesis. Galai

(1983), Rubinstein (1985) and Bossaerts and Hillion (1989) note that tests of option pricing models are known to be tests of joint hypothesis about:

1. The validity of the option pricing model
2. Correct model parameter estimation
3. Market efficiency
4. Market synchronisation
5. Data accuracy.

For reliable conclusions to be drawn about the validity of the option pricing model being examined, it is essential to have a clear formulation of the hypothesis being tested. Following Bossaerts and Hillion (1989), the joint null hypothesis employed in this study is that:

1. The option pricing model under investigation is correct
2. Markets are non-synchronous
3. The data employed in the analysis is accurate.

The issues of correct model parameter estimation and market efficiency are not included in our hypothesis. Since the set of unknown parameters that investors employ to determine the option price are implied from the model, the potential error-source of incorrect parameter estimation is eliminated. The efficiency of the options market is not under scrutiny here either, so no assumption is made about the validity of the option pricing model given the true stochastic process followed by the price of the underlying security – that is, the option pricing model need not be consistent with the true process followed by the underlying security. For example, if the market uses Black-Scholes (which assumes that the movement of the underlying security is governed by geometric Brownian motion with a constant variance) when the true process followed by the underlying security is geometric Brownian motion with a stochastic variance, the null hypothesis that the Black-Scholes model is correct will not be rejected even though the market is inefficient due to arbitrage opportunities brought about from the knowledge of the true stochastic process governing the underlying security (Bossaerts and Hillion, 1989).

Assuming the null hypothesis formulated above to be true, Bossaerts and Hillion (1989, p. 4) note that the differences between market prices and model prices are assumed 'to be generated by non-synchroneity in the observation of the contingent claims prices and the underlying security prices, and/or other security prices like the risk-free rate.' This is due to the way in which the hypothesis is constructed, i.e., we hypothesise that the option pricing model is correct and the data being employed in the study is accurate. Discounting the possibility of data inaccuracies, a rejection of the null hypothesis therefore questions the validity of the model under investigation and/or the nature of the hypothesised non-synchronous feature of the option, and associated equity-index, markets. The formulation of this hypothesis is expressed more succinctly in the following table.

Table 1: Concise Formulation of the Null Hypothesis

Definitions	Let $\mathbf{w}_{it}$ be the vector of characteristics associated with each option *i* at time *t*. This vector contains the time-to-maturity and exercise price of option *i* at time *t*. $\mathbf{w}_{it}$ may also contain additional characteristics, e.g., when the matched maturity zero-coupon yield is used as a proxy for the unobservable constant interest rate commonly assumed in option pricing models. $\mathbf{w}_{it}$ can therefore be expressed as follows: $\mathbf{w}_{it}^{0} = (\tau_{it}, K_{it}, r_{it})$ when *r* is proxied $\mathbf{w}_{it}' = (\tau_{it}, K_{it})$ when *r* is treated as an unknown constant parameter Let S_t and S_t^0 represent respectively the unobservable synchronous index price, in this instance, and the observable (i.e., the reported, or closing, price) index price at time *t*. Finally, let *f*(.) be a functional relationship between the option price, the price of the underlying security and the vector of characteristics. This functional form corresponds to the pricing model used by the market to determine option prices.

Market Price	The market price of option i at time t, which is a function of the unobservable variables S_t and r_{it} at time t, is equal to $$C_{it}^0 = f(S_t, \mathbf{w}_{it}; \mathbf{q}^*), \qquad \forall_i \in [1, m], \quad \forall_t \in [1, \hat{T}]$$ where $\mathbf{q}^*$ are the values of the parameters used by the market, m is the number of option series in the cross-section and $\hat{T}$ is the number of trading days in the sample.
Theoretical Price	The theoretical price of option i at time t, which is a function of the observable variables S_t^0 and r_{it}^0 at time t, is equal to $$C_{it} = f(S_t^0, \mathbf{w}_{it}^0; \mathbf{q}), \qquad \forall_i \in [1, m], \quad \forall_t \in [1, \hat{T}]$$ where $\mathbf{q}$ is a vector of parameters that remain constant across options and over time.
Pricing Error	From the above, it should be clear that the market is assumed to employ the same functional form $f(\cdot)$ to price options with the vector of parameters $\mathbf{q}$ set equal to $\mathbf{q}^*$. Therefore, the pricing error associated with option i at time t can be estimated as $$u_{it}(\mathbf{q}) = C_{it}^0 - C_{it}$$ $$= C_{it}^0 - f(S_t^0, \mathbf{w}_{it}^0; \mathbf{q}), \qquad \forall_i \in [1, m], \quad \forall_t \in [1, \hat{T}]$$
Hypothesis	Our hypothesis will be tested by investigating whether or not there is a statistically significant difference between Black-Scholes model prices (adjusted for dividends) of the FTSE 100 stock index-options and observed market prices. Furthermore, we will test whether there is any systematic correlation between pricing errors and the following instrumental variables: time-to-maturity, exercise price and interest rate.

In order to test precisely the hypothesis formulated above in relation to the Black and Scholes (1973) option pricing formula, it is necessary to define further what will be termed the pricing error, i.e., the difference between the theoretical price given by the formula and the market price. More formally, the pricing error is expressed as follows:

$$u_{it}^{C}(\mathbf{q}) = C_{it}^{M} - C_{it}^{T}(\mathbf{q}) \qquad \forall i \in [1,m], \forall t \in [1,\hat{T}]$$
$$u_{it}^{P}(\mathbf{q}) = P_{it}^{M} - P_{it}^{T}(\mathbf{q}) \qquad \forall i \in [1,m], \forall t \in [1,\hat{T}]$$

C_{it}^{M} = the market price of the *i*th call option at time *t*

$C_{it}^{T}(\mathbf{q})$ = the dividend-adjusted Black-Scholes model price of the *i*th call option at time *t*, evaluated at $\mathbf{q}$

P_{it}^{M} = the market price of the *i*th put option at time *t*

$P_{it}^{T}(\mathbf{q})$ = the dividend-adjusted Black-Scholes model price of the *i*th put option at time *t*, evaluated at $\mathbf{q}$.

The vector of unknown parameters, $\mathbf{q}$, includes the volatility parameter, σ, and the constant interest rate, *r*, if the constant interest rate is implied from market data. The basic concept behind GMM is to construct a family of orthogonality conditions. Therefore, the null hypothesis is expressed in terms of orthogonality/moment conditions imposed on the pricing error. An obvious condition is that the expected pricing error, evaluated at $\mathbf{q} = \mathbf{q}_0$, where $\mathbf{q}_0$ is the true value of $\mathbf{q}$, is zero, i.e.:

$$E[\mathbf{u}_t(\mathbf{q}_0)] = 0$$

Furthermore, the argument can be made that the pricing errors 'should not show a systematic relation to any variable in the investor's information set at time t, for example, the time-to-maturity of an option' (Rindell, 1994, p. 226). By letting $\mathbf{z}_t$ denote a *g*-dimensional vector of instrumental variables in the investor's information set at time *t*, the so-called unbiasedness hypothesis can be stated as follows:

$$E[\mathbf{u}_t(\mathbf{q}_0) \otimes \mathbf{z}_t] = 0$$

where $\otimes$ denotes the Kroeneker product. The Kroeneker product calls for each element of the errors vector to be multiplied by the full set of instruments in $\mathbf{z}_t$. The set of instrumental variables in this analysis are the time-to-maturity, moneyness and risk-free interest rate. To test for a particular bias, for example the time-to-maturity bias, $\mathbf{z}_t$ is set equal to the time-to-maturity instrument.

In order to implement GMM, the sample analogue for the unbiasedness hypothesis is set up as follows:

$$\mathbf{g}_\mathbf{T}(\mathbf{q};\mathbf{y}_T) = \frac{1}{T}\sum_{t=1}^{T}\mathbf{u}_t(\mathbf{q}) \otimes \mathbf{z}_t$$

$\mathbf{q}$ is chosen so as to make the sample moment $\mathbf{g}(\mathbf{q};\mathbf{y}_T)$ as close as possible to the population moment of zero; that is, the GMM estimator $\hat{\mathbf{q}}_T$ is the value of $\mathbf{q}$ that minimises the criterion function

$$Q(\mathbf{q};\mathbf{y}_T) = [\mathbf{g}(\mathbf{q};\mathbf{y}_T)]\mathbf{W}_T[\mathbf{g}(\mathbf{q};\mathbf{y}_T)]$$

whence Q (.) returns a scalar.

From the above it should be clear that $\mathbf{g}_T(\mathbf{q};\mathbf{y}_T)$, evaluated at $\mathbf{q}_0$, should be close to zero for large values of T. This has intuitive appeal as one would *expect* the pricing error/residual to be white noise, and thus show no systematic relationship (whether in cross-section or in time-series) to any instrument in the set of explanatory variables when the model value is generated by a correctly specified Black-Scholes formula. The question remains, however, as regards how well specified the Black-Scholes model is.

Goodness-of-fit Metric

In order to test the goodness-of-fit of the Black-Scholes formula against market data, and hence for the various biases specified in the instruments vector, we employ a relatively simple statistical test developed by Hansen (1982). Given that the system is overidentified, i.e., the number of orthogonality conditions exceeds the number of parameters to be estimated, the moment equations imply substantive restrictions. As such, if the hypothesis that the model that led to the moment equations in the first place is incorrect, at least some of the sample moment restrictions will be violated. In other words, we will examine whether all the sample moments represented by $\mathbf{g}(\mathbf{q};\mathbf{y}_T)$ are as close to zero as would be expected if the corresponding population moments were truly zero (Hamilton, 1994, p. 415). This provides the basis for the test (described below) of the overidentifying restrictions.

Hansen's (1982) goodness-of-fit test is based on the fact that $TQ(\mathbf{q};\mathbf{y}_T)$ is asymptotically chi-squared distributed with degrees of freedom equal to the number of overidentifying restrictions, that is:

$$TQ(\mathbf{q};\mathbf{y}_T) \rightarrow \chi^2_{r-a}$$

where r is equal to the total number of orthogonality conditions and a is equal to the number of parameters to be estimated. Chan et al. (1991) describe the test in a similar manner, noting that $TQ(\mathbf{q};\mathbf{y}_T)$ is chi-squared distributed under the null hypothesis that the model is true with degrees of freedom equal to the number of orthogonality conditions net of the number of parameters to be estimated. The goodness-of-fit hypothesis will be rejected, suggesting an incorrect model specification, when the chi-squared test statistic exceeds the chi-squared critical value read from appropriate tables.

It should be obvious, therefore, that the null hypothesis will be rejected if investors use a different model to that of Black and Scholes to price options, or if the volatility function embedded in market prices is more elaborate than the parsimonious specification peculiar to the Black-Scholes model. Indeed, such a result might be weakly construed as indirect evidence in favour of an alternative model, perhaps even a two-factor stochastic volatility model; i.e., one might infer that the market is pricing options using a two-factor stochastic volatility model as opposed to Black and Scholes (1973). Importantly, however, the rejection of the hypothesis cannot be interpreted as direct evidence in support of an alternative model. Such evidence is only admissible when such an alternative model is tested directly (Day and Lewis, 1997). A test of the aforementioned two-factor stochastic volatility model is not carried out here due to the computational burden resulting from the non-existence of an analytical solution to such a model.

DATA

The data used in this study was downloaded from a LIFFE-issued CD-ROM containing close-of-day data on all options traded on LIFFE. The particular option contract examined is the FTSE-100 ESX stock index-option contract, a European-style option with expiry months March, June, September and December, plus such additional months that the three nearest calendar months are always available for trading. All options with a bid-ask spread greater than 10 per cent are eliminated from the data set, as are all options with a time-to-maturity less than seven days. Filtering the data in this way controls for measurement error resulting from non-synchronous equity and options markets.

The time period sampled is from 1 January 1995 to 31 December 1995. The option price data consists of closing bid and ask prices that reflect the last bid and offer in the marketplace before settlement. The closing market price is proxied by taking the simple average of the bid and ask prices. The current value of the stock index can be derived from matched maturity futures prices, traded at the same market, using the cost-of-carry condition. This avoids the need for any explicit dividend adjustment as the expected dividend payments are implicitly discounted in the futures price, as explained earlier. For example, Rindell (1994, p. 227) notes that 'this procedure avoids many of the problems in using spot index values, such as non-simultaneity in individual stock prices and the correct adjustment for dividend payments.'

Rindell (1994, p. 226) notes that 'in nearly all empirical tests, with the exception of Bossaerts and Hillion (1989), the constant interest rate is proxied with the yield of a matched maturity zero-coupon bond.' Due to the unavailability of 6-month and 12-month treasury bill data for the time period considered, we proxy the risk-free interest rate parameter by interpolating the 1-month and 3-month discount treasury bills, and assuming a flat term structure for maturities above three months. This interest rate data was downloaded from Datastream/ICV.

RESULTS AND ANALYSIS

The following orthogonality conditions are tested:

(i) $E[\mathbf{u}_t] = \mathbf{0}$ (ii) $E[\mathbf{u}_t \tau_t] = \mathbf{0}$

(iii) $E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = \mathbf{0}$ (iv) $E[\mathbf{u}_t \mathbf{r}_t] = \mathbf{0}$

where $\mathbf{u}_t$, for example, is an $r=mg$-dimensioned vector of pricing errors, and $\mathbf{0}$ is an r-dimensioned null vector. The last three orthogonality conditions can thus be interpreted as empirical tests of the time-to-maturity bias, the moneyness bias and the interest rate bias. In our empirical analysis, the extent to which an option is in the money is measured as the natural logarithm of the ratio between the futures price of the index and the strike price.

The results of the various tests are reported above. Each row gives detailed results for the orthogonality conditions specified in the first column. The parameter estimates and the corresponding t-statistics are given in the second and third columns respectively. Finally, the chi-squared test statistic and corresponding degrees of freedom and p-values are reported in the last three columns respectively.

When all the orthogonality conditions are imposed simultaneously the model is strongly rejected. Rindell (1994, p. 228) notes, however, that 'since a model only approximates reality, it will always be rejected if the test is powerful enough.' A more appropriate question, therefore, is *why* the model is rejected. This question is answered by examining the various different combinations of the orthogonality conditions. The table shows that the Black and Scholes (1973) model is not rejected when the orthogonality conditions are tested individually. The number of degrees of freedom is equal to four since there are five distinct moment conditions and only one parameter, the volatility, being estimated. The null hypothesis that

$$E[\mathbf{u}_t \mathbf{z}_t] = \mathbf{0}$$

cannot be rejected. In other words, the hypothesis that the pricing error is equal to zero on average, or is uncorrelated with the time-to-maturity, moneyness, or interest rate cannot be rejected by the data. One could conclude from this first set of results that the option pricing model under investigation does not seem to display any particular bias. The next stage of the testing procedure is to increase the power of the test by imposing the orthogonality conditions in pairs.

Table 2: GMM Tests of the Black and Scholes (1973) Model: Implying Volatility Only

	Parameter estimates		Goodness-of-fit		
Orthogonality Conditions	σ	*t-value*	χ^2 computed	*DF*	*p-value*
$E[\mathbf{u}_t] = 0$	0.1414	178.97	2.738	4	0.6026
$E[\mathbf{u}_t \tau_t] = 0$	0.1447	185.63	2.065	4	0.7237
$E[\mathbf{u}_t \mathbf{r}_t] = 0$	0.1413	181.11	2.895	4	0.5756
$E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.1430	180.03	2.374	4	0.6674
$E[\mathbf{u}_t] = E[\mathbf{u}_t \tau_t] = 0$	0.1438	186.68	99.296	9	**<.000**
$E[\mathbf{u}_t] = E[\mathbf{u}_t \mathbf{r}_t] = 0$	0.1397	196.53	24.988	9	**0.0030**
$E[\mathbf{u}_t] = E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.1385	180.78	153.47	9	**<.000**
$E[\mathbf{u}_t \tau_t] = E[\mathbf{u}_t \mathbf{r}_t] = 0$	0.1434	187.09	101.58	9	**<.000**
$E[\mathbf{u}_t \tau_t] = E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.1406	188.39	159.89	9	**<.000**
$E[\mathbf{u}_t \mathbf{r}_t] = E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.1384	183.52	153.09	9	**<.000**
$E[\mathbf{u}_t] = E[\mathbf{u}_t \tau_t] = E[\mathbf{u}_t \mathbf{r}_t] = 0$	0.1427	201.69	108.95	14	**<.000**
$E[\mathbf{u}_t] = E[\mathbf{u}_t \tau_t] = E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.1399	189.33	168.68	14	**<.000**
$E[\mathbf{u}_t] = E[\mathbf{u}_t \mathbf{r}_t] = E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.1370	203.70	152.28	14	**<.000**
$E[\mathbf{u}_t \tau_t] = E[\mathbf{u}_t \mathbf{r}_t] = E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.1394	190.03	168.08	14	**<.000**
$E[\mathbf{u}_t] = E[\mathbf{u}_t \tau_t] = E[\mathbf{u}_t \mathbf{r}_t] = E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.1383	216.52	166.27	19	**<.000**

Six distinct pairwise combinations of moment conditions are obtained when the orthogonality conditions are imposed in pairs. The number of degrees of freedom is nine, since 10 moment conditions are tested and again only one parameter estimated. The results of these tests are significantly different from the first set in two distinct ways. Firstly, the chi-squared test statistic increases substantially in all cases causing the moment conditions to be rejected at the one per cent level of significance, i.e., the goodness-of-fit of the model under investigation disimproves. Another more subtle difference is that the chi-squared test statistic obtained by testing the first moment and any cross-moment conditions simultaneously is always higher than the sum of the chi-squared test statistics obtained by testing the two moment conditions separately. Bossaerts and Hillion (1989) interpret similar results as evidence that the biases are probably not correlated.

The moment conditions are then tested in sets of three. This results in four distinct triplets of moment conditions. Now the number of degrees of freedom is equal to 14. As one would expect, each triplet of moment conditions is rejected at a very high level of significance. An interesting aberration here is how the chi-squared test statistic for the unit instrument, the moneyness and the interest rate is in fact lower than the chi-squared test statistic for the unit instrument and the moneyness tested as a pair. This, along with some of the other results, may indicate that the interest bias associated with the Black-Scholes model may not be very strong. Observe that when the interest rate is tested in a pair along with the unit instrument it has a relatively low (although significant) chi-squared test statistic. When either the moneyness instrument or the time-to-maturity instrument is added to make a triplet, the chi-squared test statistic rises significantly. However, when the interest rate is added to any of the other pairs to make a triplet, it does not have the same effect.

The estimates of the unknown parameter are relatively stable across the different orthogonality conditions. They are generally approximately equal to 14 per cent and have high *t*-values, indicating that the standard errors of the estimates are low. This estimate compares favourably with the historical time-series estimate for the same period of the variance of the logarithm of stock price returns, which equals approximately 12 per cent, and with the average implied volatility figure calculated by Datastream/ICV which is approximately 13 per cent.

The empirical results reported in the above table can be summarised as follows. The Black-Scholes model is not rejected when the moment conditions are tested individually, regardless of what instrumental variable is chosen. However, when the moment conditions are tested in pairs and higher combinations, the model is always rejected. There is some evidence suggesting that the interest rate bias may not be as strong as the moneyness and time-to-maturity biases. Another feature of the results is how stable the volatility estimate is across the moment conditions, varying tightly around the 14 per cent mark. Finally, the increasing power of the test as the number of moment equations increases is evident from the increasing chi-squared statistic values. These results are strongly consistent with those of Rindell (1994) and Bossaerts and Hillion (1989).

GMM Tests of the Black and Scholes (1973) Model: Implying Volatility and Interest Rate from the Model

The use of a proxy for the unobservable constant continuous time interest rate is one possible reason for the rejection of the Black-Scholes model as outlined in the above analysis. Another explanation might be that non-sychroneity is systematic rather than non-systematic. Bossaerts and Hillion (1989) note that there are several reasons to suspect non-synchroneity to be systematic. For example, non-synchroneity is likely to be systematic if option prices are systematically observed with a time lag with respect to the underlying security price. In such cases, if the underlying security experiences a non-zero price change during the *observation lag*, it is likely to cause a non-zero mean non-sychroneity error.

The interest rate instrument is examined below by implying the interest rate from the model to determine whether the use of a proxy has any impact on the rejection of the model. Unlike the first set of results, where the volatility is the only unknown parameter implied from the data, the following tests are conducted by letting the vector of unknown parameters contain the interest rate parameter as well as the volatility parameter. **Table 3** summarises the results. Note that the parameter estimates and their associated *t*-values are reported in columns two through five. The results in relation to the goodness-of-fit hypothesis are again contained in the final three columns. Note that a degree of freedom is lost in these tests, due to the fact that two parameters are implied from the model being tested.

Table 3: GMM Tests of the B-S (1973) Model: Implying both Volatility and Interest Rate

	Parameter Estimates				Goodness-of-fit		
Orthogonality Conditions	σ	*t-value*	r	*t-value*	χ^2 computed	*DF*	*p-value*
$E[\mathbf{u}_t] = 0$	0.145	20.22	0.073	3.68	1.927	3	0.5877
$E[\mathbf{u}_t \tau_t] = 0$	0.147	17.73	0.069	3.45	1.769	3	0.6210
$E[\mathbf{u}_t \mathbf{r}_t] = 0$	0.144	20.51	0.073	3.72	2.107	3	0.5505
$E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.153	6.978	0.037	14.06	2.166	3	0.5386
$E[\mathbf{u}_t] = E[\mathbf{u}_t \tau_t] = 0$	0.122	49.46	0.031	2.20	9.906	8	0.2717
$E[\mathbf{u}_t] = E[\mathbf{u}_t \mathbf{r}_t] = 0$	0.138	36.05	0.061	5.94	29.757	8	**0.0002**
$E[\mathbf{u}_t] = E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.132	159.13	0.038	21.61	9.652	8	0.2903
$E[\mathbf{u}_t \tau_t] = E[\mathbf{u}_t \mathbf{r}_t] = 0$	0.121	47.90	0.011	1.80	10.186	8	0.2522
$E[\mathbf{u}_t \tau_t] = E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.133	150.69	0.038	20.77	8.657	8	0.3720
$E[\mathbf{u}_t \mathbf{r}_t] = E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.132	160.96	0.038	21.67	9.690	8	0.2873
$E[\mathbf{u}_t] = E[\mathbf{u}_t \tau_t] = E[\mathbf{u}_t \mathbf{r}_t] = 0$	0.121	51.80	0.015	2.65	33.272	13	**0.0015**
$E[\mathbf{u}_t] = E[\mathbf{u}_t \tau_t] = E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.132	156.33	0.034	19.35	41.030	13	**<.000**
$E[\mathbf{u}_t] = E[\mathbf{u}_t \mathbf{r}_t] = E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.130	173.43	0.039	22.45	36.466	13	**<.000**
$E[\mathbf{u}_t \tau_t] = E[\mathbf{u}_t \mathbf{r}_t] = E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.131	158.27	0.034	19.23	44.340	13	**<.000**
$E[\mathbf{u}_t] = E[\mathbf{u}_t \tau_t] = E[\mathbf{u}_t \mathbf{r}_t] = E[\mathbf{u}_t \ln(\mathbf{F}_t / \mathbf{K}_t)] = 0$	0.130	171.29	0.035	20.07	59.630	18	**<.000**

As one would expect, the goodness-of-fit is improved in all bar one case, the only exception being the orthogonality condition where the interest rate and the unit instrument are combined.

When the moment conditions are imposed individually, we again fail to reject the model in all cases. This is consistent with the results of the first test. However, in contrast to the first test when the moment conditions are tested in pairs we again fail to reject the model except for the case where the unit instrument and the interest rate are combined. The Black-Scholes model is rejected in all cases when the moment conditions are imposed three by three. As one would expect, the model is also rejected when all of the moment conditions are imposed simultaneously. A possible explanation for these results may be that the constant continuous time interest rate Black-Scholes model is a better fit than the version that uses a proxy for the interest rate, or that the market uses a constant interest rate rather than the proxy.

Another very important factor and possible explanation for these conflicting results lies in the fact that an element of non-simultaneity is eliminated when the interest rate is implied from that data rather than trying to proxy it. Bossaerts and Hillion (1989, p. 27) note that 'the treatment of *r [the interest rate]* as an unknown parameter yields tests that are virtually free of the interest rate induced non-synchroneity'. The non-synchroneity error arises because it is almost impossible to find a bond that matures on exactly the same day and at the same time as the option.

The interest rate implied from the model produces plausible figures in most cases. The *t*-statistics are highest when the interest rate is estimated to be around 4 per cent. There are estimates that range as high as 7.39 per cent and as low as 1.11 per cent. In relation to the volatility estimates produced under this test, they remain at approximately 14 per cent, although they do show more variation than the first test. It should be remembered in one's interpretation of this that GMM is a best-fit procedure and, given that two parameters are involved in this test, the increased variation is not surprising.

The results above confirm the existence of biases in the Black and Scholes (1973) option pricing model as applied to LIFFE ESX FTSE 100 stock index-options. The time-to-maturity and moneyness instruments have been extensively studied in the literature and a high degree of correlation between them and option pricing errors has been noted. Hence the now familiar terms, time-to-maturity bias and exercise price or moneyness bias. Bossaerts and Hillion (1989, p. 21) note, however, that these studies are to a large extent inconclusive, 'since the biases are tested using ordinary *t*-statistics which fail to take into account obvious properties of the pricing errors such as errors-in-variables and heteroscedasticity'. Also, these studies have used linear regressions on variables that are non-linearly related. Therefore, the results above are more robust evidence confirming the existence of the aforementioned biases.

Specifically, GMM tests are carried out on the Black-Scholes (1973) option pricing model adjusted for dividends. The GMM tests reject the model, which is underpinned by the assumption that volatility and interest rates are constant. The well-known moneyness bias and the time-to-maturity bias are both seen to be significant. So too is the interest rate bias, but it seems that it may not be as significant as the two other biases. Since the potential problem of non-synchronous data has been mitigated, we might infer that investors use a different option pricing model to the one being investigated. Furthermore, one might interpret the empirical results as evidence that the volatility function, or process, embedded in the LIFFE ESX option prices is not consistent with the underlying assumption of a constant volatility parameter in the Black-Scholes formula, and therefore conclude that investors use an alternative option pricing model that allows for either (i) a more elaborate deterministic volatility specification, or (ii) an autonomous stochastic volatility process.

SUMMARY DISCUSSION AND CONCLUSIONS

This paper implements a GMM-based empirical test of the Black and Scholes (1973) model for the valuation of European-style call options on LIFFE. Since GMM can be seen as a generalised non-linear instrumental variables procedure, it is particularly suited to test contingent claims asset pricing models, as these models are non-linear functions of the parameters in the pricing formulas (Bossaerts and Hillion, 1989). The use of GMM allows us to (i) assess the statistical significance of the pricing error, (ii) test for the presence of biases, (iii) estimate unobservable parameters of the option pricing model, and (iv) report heteroscedasticity-consistent standard errors of the parameter estimates. The results confirm the findings of most other previous studies of the model, i.e., the model is biased with respect to time-to-maturity, moneyness and, possibly to a lesser extent, the risk-free interest rate.

If the Black-Scholes model were truly correctly specified, then the error term should not conform to any systematic pattern of occurrence either in cross-section or in time-series. In reality, the observed moneyness, time-to-maturity and interest rate biases of Black-Scholes are evidence that this is not the case, whence the Black-Scholes model must be misspecified relative to the chi-squared goodness-of-fit metric. Further, the constant variance assumption is the most obvious source of model misspecification, given (i) that we have adjusted the model for a dividend correction, (ii) the controls we have used for non-synchronous measurement error, and (iii) the body of empirical evidence that exists as *prima facie* support for the autonomous mean-reverting stochastic character of stock return volatility over time.

The weight of empirical evidence in support of the stochastic mean-reverting character of stock return volatility over time is considerable. Many empirical studies undermine the validity of the Black-Scholes constant-volatility assumption (Black and Scholes, 1972; Latane and Rendleman, 1976; Schmalensee and Trippi, 1978). In response to such studies, several authors developed models that allow the volatility of stock prices to change as a function of the underlying stock price (Cox, 1975; Geske, 1979). However, these models only partially explain movements in the volatility parameter (Emanuel and Macbeth, 1982). Accordingly, authors such as Hull and White (1987), Johnson and Shanno (1987), Scott (1987) and Wiggins (1987) developed option pricing models that describe the volatility parameter itself as an autonomous random variable that satisfies some stochastic differential equation.

Therefore, we might intuitively reason that our results strongly reject the Black-Scholes model, because it may not be the model that the market is using to price options. It is obvious from the literature that the Black-Scholes model has been extended upon and improved since its inception in 1973, e.g., constant elasticity of variance models, implied volatilities as opposed to a constant volatility parameter, stochastic volatility models, etc. Therefore, it is quite possible that the market is using more advanced and empirically consistent formulas such as the two-factor stochastic volatility models mentioned above. This may also explain how our results differ from the Bossaerts and Hillion (1989) analysis, which is based on a futures-options commodity market where the Black (1976) formula has been much favoured by market practitioners in determining market prices of commodity claims.

As suggested, the rejection of the constant-variance Black-Scholes model using the Hansen GMM goodness-of-fit metric is most likely due to the fact that market participants in the index options market are using a different model in determining market prices, ranging from possibly a more elaborate functional form for the volatility parameter to more complex and empirically-consistent jump-diffusion cum stochastic volatility models. In the latter models, or indeed where both sources of uncertainty are combined (Bates, 1996), the completeness property of the Black-Scholes model no longer obtains, whence pricing via an absence of arbitrage argument is not feasible and an equilibrium pricing approach is called for.

The equilibrium (as opposed to arbitrage-free) option-pricing literature has been dominated by the partial equilibrium paradigm in which Arbitrage Pricing Theory (APT)-style models characterise the equilibrium expected excess return on the option as being the (possible) sum of a number of utility-dependent risk-premia, corresponding in number to the number of APT-type risk factors posited in the particular model, e.g., two in the case of a stochastic volatility model. Although appearing to be theoretically intuitive as regards consistency with the APT model, these models require facilitative market-equilibrium assumptions – some of which can be questionable to say the least – to be analytically tractable, and capable of empirical testing (e.g., the closed-form solution in the Hull and White (1987) diversifiable random volatility risk model).

A fruitful extension of the theoretical-cum-empirical GMM framework of this paper might be to look at a mixed jump-diffusion model (or indeed a jump-diffusion cum stochastic volatility model) in which the underlying risk-source is systematic/undiversifiable. Bates (1996) has recently used such a model to recover the implicit distributional characteristics of jump-sensitive currency and index options, by way of accounting for the excess kurtosis and skewness effects that are characteristic of the empirical processes in these markets, and which are clearly not consistent with the symmetric lognormal distributional assumption of the Black-Scholes model. We believe that implementation of our GMM estimator would be new in this context. Importantly, the multi-dimensional estimation approach would permit the recovery of a vector of model parameters, which could then serve as the inputs in the construction of conditional skewness and kurtosis plots for options of various maturity. The latter provide important empirical insights in terms of a model's ability to account for well-known distributional anomalies associated with the Black-Scholes model.

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